Geometric Series:

Suppose you wanted to find the partial sum: $S_n = 4 + 12 + 36 + \cdots + 236196$

Method 1: ❶ Figure out all of the missing numbers ❷ Add them all up without making mistakes

Method 2: Use a formula

Let's start with an easier arithmetic partial sum: $S_n = 4 + 12 + 36 + 108 + 324 =$ 484

The formula for this one is harder to derive.

$$S_n = 4 + 12 + 36 + 108 + 324$$

$$3 \cdot S_n = \quad 12 + 36 + 108 + 324 + 972$$

$$1 - 3 \cdot S_n = 4 \qquad\qquad - 972$$

So $(1-3)S_n = 4 - 972 \rightarrow S_n = \frac{4-972}{1-3} = \frac{-968}{-2} = 484$

In this example, $4 = a_1$, $3 = r$, $972 = 3 \cdot 324 = r \cdot a_n$

So the formula for the partial sum of a geometric series is: $S_n = \frac{a_1 - r \cdot a_n}{1-r}$

An alternate formula is: $S_n = \frac{a_1(1-r^n)}{1-r}$

Ex: Find $S_n = 1 + 3 + 9 + 27 + 81 + 243 + 729$

$a_1 = 1,\quad a_n = 729,\quad r = 3,\quad n = 7$ ❯ $S_n = \frac{1-3(729)}{1-3}$ OR $\frac{1(1-3^7)}{1-3} = \frac{-2186}{-2} = -1093$

Ex: Find

$$S_n = \sum_{n=1}^{15} 3 \cdot 4^{n-1}$$

$a_1 = 3,\quad r = 3,\quad n = 15$ ❯ $S_n = \frac{3(1-3^{15})}{1-3} = \frac{3(-14348906)}{-2} = \frac{-43046718}{-2} = 21523359$

Ex: Find $S_n = 2 + 4 + 8 + \cdots + 8388608$

$a_1 = 2,\quad r = 2,\quad n = ?,\quad a_n = 8388608$ ❯ $S_n = \frac{2-2 \cdot 8388608}{1-2} = \frac{2-16777216}{-1} = \frac{-16777214}{-1} = 16777214$

What about an infinite geometric series: $S_n = 2 + 4 + 6 + 8 + \cdots =$ bigger and bigger = NO SUM

What if you are multiplying by a small number? (each term gets smaller) $S_n = \frac{a_1}{1-r}$ only if $|r| < 1$

Ex: $S_n = 180 + 18 + 1.8 + 0.18 + \cdots$ ❯ $a_1 = 180,\quad r = 0.1$ ❯ $S_n = \frac{180}{1-0.1} = \frac{180}{0.9} = 200$